ON ELASTICITY IN DYNAMIC CABLE EQUATIONS

by

D.M. Stump
School of Civil and Mining Engineering
The University of Sydney
NSW 2006 Australia

Cable equations have a long history and have been extensively studied by many authors. The nonlinear static equations describing perfectly flexible and inextensible cables under gravity loading are a common feature of elementary calculus courses, since closed-form catenary solutions exist. For more complicated geometries and loadings, this is not the case and even the static equations must be integrated numerically.

Of considerable importance is the issue of vibration in cables. As an example, consider electricity transmission cables, where the continuous buffeting by wind gusts induces transient waves which contribute to fatigue at the fixture points to the tower. Clearly, for this case, an understanding of transient phenomenon may aid in reduction of fretting and reduced failures through improved designs.

Transient solutions to cable equations have been studied primarily as small perturbations to static solutions. Much recent progress has come from Caughy and coworkers at CalTech (e.g. Caughy and Irvine, 1974), who have examined the small vibration of elastic cables. More recently Watts and Frith (1981) have considered the numerical solution of the nonlinear equations governing the large vibrations of an initially straight and taut elastic cable.

Elastic cables contain two separate and distinct time scales: the longer transient sway time of the cable as a pendulum, and the typically much faster time for longitudinal elastic waves to run back and forth along the cable. Most previous studies, including the ones cited above, have been concerned with the transient behavior on the longer time-scale and have not considered the interactions with the much faster elastic waves. It has been tacitly assumed that the magnitude of elastic displacements must be small in comparison to motion of the cable due to loadings applied along the length of the cable. But elastic forces can be very large and it should not be assumed automatically that the tension associated with small elastic motions is minuscule in comparison to that arising from the gross cable motion.

This note presents some preliminary results from a new numerical scheme that is capable of integrating the complete nonlinear equations of motion. The time-stepping is fully implicit, and so is unconditionally stable, and the technique is robust enough to resolve the fine detail necessary to observe the elastic waves.

A coordinate system s is engraved on the unstretched cable with the initial coordinate length s_l. The physical length of the loaded cable will change due to elasticity but the coordinate system is fixed at length s_l. The positions and tension of the cable are described by $\mathbf{R}(s,t)$ and $T(s,t)$ as functions of s and time t. The cable spans a horizontal distance l along the x-axis of a coordinate system and two forces are in action: gravity acting vertically downward, and a velocity squared air-drag acting in the direction of the velocity normal to the cable. The cable is extensible with the modulus EA, where E is the Young's modulus and A is the cross-sectional area of the unloaded cable.

The equations of motion of the cable are given by

$$m\frac{\partial^2 \mathbf{R}}{\partial t^2} = \frac{\partial}{\partial s}\left(T\frac{\partial \mathbf{R}}{\partial s}\right) + \frac{T^2}{EA}\frac{\partial^2 \mathbf{R}}{\partial s^2} + \mathbf{F} \tag{1}$$

where: the applied force $\mathbf{F}$ is

$$\mathbf{F} = -mg\mathbf{k} + D_n\left|\mathbf{v}_n\right|\left(1 + \frac{T}{EA}\right)\mathbf{v}_n, \tag{2}$$

m is the linear mass density, and D_n is an air-drag coefficient. The normal velocity $\mathbf{v}_n$ is

$$\mathbf{v}_n = \mathbf{v}_R - \left(\frac{\partial \mathbf{R}}{\partial s}\cdot\mathbf{v}_R\right)\frac{\partial \mathbf{R}}{\partial s}\left(1 + \frac{T}{EA}\right)^{-2} \tag{3}$$

and the velocity of the cable relative to the applied gust $\mathbf{V}$ is

$$\mathbf{v}_R = \mathbf{V} - \frac{\partial \mathbf{R}}{\partial t}. \tag{4}$$

The extensibility of the cable is given by

$$\frac{\partial \mathbf{R}}{\partial s}\cdot\frac{\partial \mathbf{R}}{\partial s} = \left(1 + \frac{T}{EA}\right)^2 \tag{5}$$

but a more useful form can be obtained by forming the dot product of $\frac{\partial \mathbf{R}}{\partial s}$ and (1).

Equations (1) and (5) constitute two nonlinear partial differential equations which can be nondimensionalized using the span l and the sway time $\tau_s = \sqrt{l/g}$. This selection leads to dimensionless (barred) quantities

$$\left.\begin{array}{l} \bar{t} = \frac{t}{\tau_s}, \quad \bar{\mathbf{R}} = \frac{\mathbf{R}}{a} = \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}, \\ \bar{s} = \frac{s}{l}, \quad \bar{\mathbf{v}}_n = \frac{\mathbf{v}_n}{\sqrt{gl}}, \quad \bar{\mathbf{v}}_R = \frac{\mathbf{v}_R}{\sqrt{gl}}, \\ \bar{T} = \frac{T}{mgl}, \quad \bar{\mathbf{F}} = \frac{\mathbf{F}}{mg} \end{array}\right\} \tag{6}$$

and the elasticity parameter

$$\epsilon = \frac{mgl}{EA} \tag{7}$$

which is related to the ratio of the characteristic times for the vibrational and transverse motions,

$$\frac{\tau_v}{\tau_s} = \sqrt{\frac{ml^2/EA}{l/g}} = \sqrt{\epsilon}. \tag{8}$$

Dropping the bar notation, the dimensionless equations are rewritten

$$\frac{\partial^2 \mathbf{R}}{\partial t^2} = \frac{\partial}{\partial s}\left(T\frac{\partial \mathbf{R}}{\partial s}\right) + \epsilon T^2 \frac{\partial^2 \mathbf{R}}{\partial s^2} + \mathbf{F} \tag{9}$$

and

$$\frac{\partial \mathbf{R}}{\partial s} \cdot \left\{\frac{\partial^2 \mathbf{R}}{\partial t^2} - \mathbf{F}\right\} = (1+\epsilon T)\frac{\partial T}{\partial s}, \tag{10}$$

and the fixed boundary conditions are given by $\mathbf{R}(0,t) = \mathbf{0}$ and $\mathbf{R}(s_l,t) = \mathbf{e}_x$.

The solution to the static cable equations can be expressed as a perturbation series in the small parameter ϵ. Attempting to extract the elasticity from the dynamic equations via a similar method fails, however, due to the presence of phenomenon on multiple time and length scales. The procedure of multiscaling has, so far, not been successful in ascertaining the appropriate form of an expansion, and the full nonlinear equations must be integrated directly for specific ϵ's.

Calculations are performed by replacing all partial derivatives with respect to time by backwards difference operators and then solving sequentially a series of spatial problems by the relaxation method (Press et al., 1986). Stump and Fraser (1994) have used this method to integrate the inextensible cable equations that arise during the wool manufacturing process of ring spinning and have obtained rapid solution.

The importance of elasticity in the equations is illustrated in the following example. Consider a cable that is hanging under gravity and a steady crosswind. At time $t = 0$ the wind is increased by 10% and it is desired to calculate the transient behavior as the cable moves into it's new configuration. For example purposes, we use the parameter values

$$s_l = 2, \ \epsilon = 0.001, \ D_n = 1.0, \ \mathbf{V} = 1.0\mathbf{j}$$

(Assuming a value of $l = 30$ meters, the magnitude of $\mathbf{V}$ corresponds to a sustained wind of 62 km/hour. The value of D_n is reasonable given the model of a cylinder in turbulent flow.) The steady-state profile of the cable is essentially identical to the results for the inextensible model (i.e. $\epsilon = 0$).

At time $t = 0$, $\mathbf{V} = 1.1\mathbf{j}$ is applied and the computational procedure is used to calculate the response. In order to adequately capture the effects of elasticity a large number of spatial dicretization points and small time steps are required. Spatial and temporal phenomena occur on the scales of order $s/\sqrt{\epsilon}$ and $t/\sqrt{\epsilon}$ and the grid in domain of (s,t) space must be refined in comparison to these dimensions. For this problem, 750 spatial points along the cable of length $s_l = 2$ were used and several different time steps were tried.

Figure 1 shows the percentage increase in the fixture-point tension $T(0,t)$ with respect to the initial steady-state value as a function of time. Calculations were performed using time steps of $\delta t = 0.001$, 0.005 and 0.01. The fractional change in tension over the interval $0 \le t \le 10$ is shown in figure 1a for the largest time step and clear shows the damped sway of the cable on the order of τ_s. Figure 2 shows the shorter time response over $0 \le t \le 2$ for all three time steps. The effects of elasticity are significant in the initial response and cause a series of large tension waves to impact the boundary.

Results for other parameter values and different magnitude wind gusts have shown similar effects and confirm that the presence of elasticity can cause a decaying oscillatory behavior that is not predicted by the inextensible model and which may be significant in promoting fatigue since the initial magnitude of the spikes is not small.

In summary, the effects of elasticity are intrinsic to the nonlinear equations of motion and linearization schemes have not been successful in extracting the behavior while retaining the essential phenomena. This preliminary work shows that transient tension waves due to elasticity may be much larger and have more important consequences than has been previously appreciated.

REFERENCES

H.M. Irvine and T.K. Caughy (1974), "The linear theory of free vibrations of a suspended cable", *Proc. Roy. Soc. Lond. A*, **341**, pp 299-315.

A.M. Watts and R.H. Frith (1981), "Efficient numerical solution of the dynamic equations of cables", *Comp. Meth. in App. Mech. and Eng.*, **25**, pp 1-9.

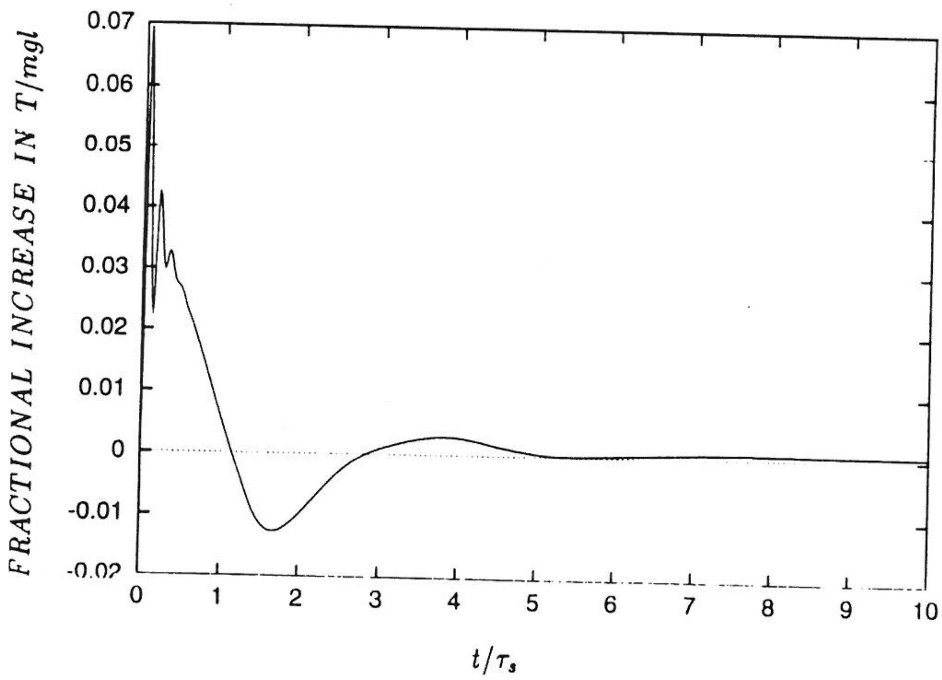

Figure 1. Plot of fractional increase in tension versus time for $\delta t = 0.01$.

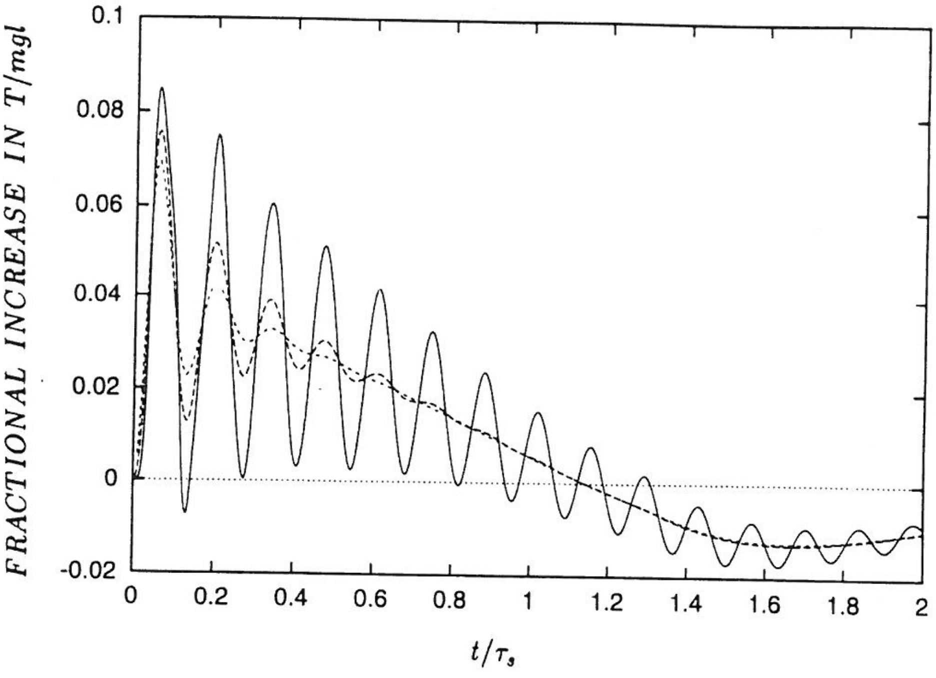

Figure 2. Plot of fractional increase in tension versus time for $\delta t = 0.001$ (solid line), $\delta t = 0.005$ (long dash line), and $\delta t = 0.01$ (short dash line).